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DISPERSION RELATIONS AND MODE SHAPES FOR WAVES IN LAMINATED VISCOELASTIC COMPOSITES BY VARIATIONAL METHODS

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Abstract—The propagation of oscillatory waves through periodic elastic composites has been analysed on the basis of the Floquet theory. This leads to self-adjoint differential equation systems which it was proved convenient to solve by variational methods. Many composites, such as the light-weight high-strength boron-epoxy material, consist of strong reinforcing components in a plastic matrix. The latter can exhibit viscoelastic properties which can have a significant influence on wave propagation characteristics. Replacement of the elastic constant by the viscoelastic complex modulus changes the mathematical structure so that the differential equation system is no longer self-adjoint. However, a modification of the variational principles is suggested which retains formal self-adjointness, and yields variational principles which contain additional boundary terms. These are applied to the determination of wave speeds and mode shapes for a laminated composite made of homogeneous elastic reinforcing plates in a homogeneous viscoelastic matrix for plane waves propagating normally to the reinforcing plates. These results agree well with the exact solution which can be evaluated in this simple case. The variational principles permit solutions for periodic, but otherwise arbitrary variation of material properties.

INTRODUCTION

A composite medium typically consists of a matrix material with an embedded reinforcing material in the form of fibers or laminations. A laminated composite can usually be modelled as alternating layers of matrix and filament arranged in a periodic manner, while a fiber reinforced composite can often be represented by a homogeneous matrix in which a two dimensional doubly periodic array of filaments is embedded.

A convenient method for analysing wave propagation in composite media is the use of Floquet theory. This approach has been recently used by several authors[1–7] to study steady state wave propagation in periodic elastic composites. Of these, some[1–5] used variational methods while others[6, 7] used direct numerical methods involving discretization of the governing differential equations and associated quasi-periodic boundary conditions.

In practice, the matrix in a composite is often a polymer which exhibits viscoelastic properties. This leads to dissipation as well as dispersion. Free wave propagation in an infinite laminated viscoelastic composite was studied by the present authors in [8] using Floquet theory and finite difference methods. In the viscoelastic case, forced steady wave propagation leads to real frequencies and complex wave numbers (attenuating waves) while free waves with no applied tractions have complex frequencies and real wave numbers (damped waves)[8].

Consideration of viscoelastic properties of the matrix leads to complex viscoelastic moduli which are functions of the frequency that can be real or complex. Even in the elastic case[1] it is convenient to utilize complex analysis to incorporate the changing phase as the wave traverses the periodic medium, so that the displacement is expressed by

$$\bar{u} = u(x)\,e^{i\omega t}$$

where ω is the frequency, t is time and $u(x)$ is complex. Application of variational methods involves integrals of the type

$$I(u_1, u_2) = \int_{-a/2}^{a/2} \left\{ -\eta(x)\frac{du_1^\dagger}{dx}\frac{du_2}{dx} + \rho(x)\omega^2 u_1^\dagger u_2 \right\} dx$$

where $\eta(x)$ is the elastic modulus variation, $\rho(x)$ that of the density, a is the periodic cell length and * denotes the complex conjugate. For elasticity with η and frequency ω real, the integrals are Hermitian: $I(u_1, u_2) = I^*(u_2, u_1)$ and self-adjoint differential operator systems result for determining the displacement or stress fields. For viscoelasticity, when η and ω are complex, the integrals are not Hermitian since $\eta^* \neq \eta$ and the resulting differential operators are non-self-adjoint. However, if one formally bases the theory on real analysis, so that the complex conjugate sign is removed from the integral even though $u(x)$ is complex, the integral is then symmetric: $I(u_1, u_2) = I(u_2, u_1)$, and hence formally self-adjoint differential operators are generated. This is so, with $u(x)$ complex, even though the integral $I(u, u)$ is then no longer physically an energy integral and the variational principle no longer expresses Hamilton's principle. This formulation introduces an additional boundary term, so that the system is formally self adjoint: a self-adjoint operator but not with the appropriate boundary conditions. These boundary terms can be included to generate variational principles, and the ones presented here are formulated in this way.

In this paper, the strain energy[1], complementary energy[2] and Hellinger-Reissner[3–5] variational principles are extended so that they can be used in the viscoelastic case. As an illustration, the extended version of the Hellinger-Reissner[4] variational principle is applied to the problem of free waves in an infinite one dimensional viscoelastic composite[8]. This principle is chosen because it gave very accurate results in the elastic case[4]. A composite with two homogeneous layers per cell is studied with the filament elastic and the matrix modelled as a three element solid. The Rayleigh-Ritz method is used to obtain dispersion relations and mode shapes and the results compared with the exact solution which exists in this case. For two homogeneous components the elastic solution has been obtained in closed form[9, 2] and this can be adapted to the case of linear viscoelastic components. The question of rapidity of convergence to the exact solution is discussed.

The present method can be used to study wave motions in general one-dimensional periodic viscoelastic composites which otherwise exhibit arbitrary variations of material properties. Waves in fiber reinforced viscoelastic composites can be studied by using similar variational principles in two dimensions.

GOVERNING EQUATIONS

We consider wave propagation in a one dimensional laminated viscoelastic composite, and, in particular, choose a two material composite as shown in Fig. 1. This example is chosen because the variational principles to be presented in the next section can then be easily compared with those in the elastic case[1–5]. Variational principles for any general one-dimensional inhomogeneous periodic viscoelastic medium can be easily obtained by suitable modification of these principles.

The composite covers the full space $-\infty < x < \infty$. We study one-dimensional strain waves propagating in a direction x normal to the interface planes. For harmonic waves, the stress and displacement are of the form $\bar{\sigma} = \sigma(x)e^{i\omega t}$ and $\bar{u} = u(x)e^{i\omega t}$. The quantities σ and u are, in general, complex.

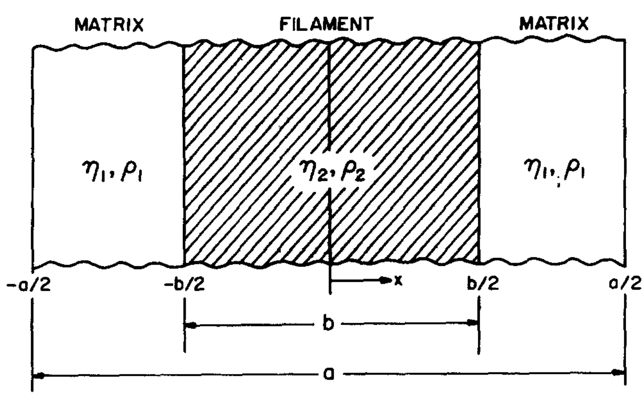

Fig. 1. Composite cell.

Within each cell, the governing differential equations are

$$\frac{d\sigma}{dx} + \rho(x)\omega^2 u = 0 \tag{1}$$

$$\sigma = \eta(x, \omega)\frac{du}{dx} \tag{2}$$

where $\eta = \lambda + 2\mu$ in terms of complex Lamé viscoelastic moduli of the constituent materials. The quantities η and ρ are periodic with period a

$$\eta(x+a;\omega) = \eta(x;\omega), \quad \rho(x+a) = \rho(x)$$

and within each cell they are discontinuous functions defined by

$$\eta(x;\omega), \rho(x) = \begin{cases} \eta_1(\omega), \rho_1 & -a/2 < x < -b/2 \\ \eta_2(\omega), \rho_2 & -b/2 < x < b/2 \\ \eta_1(\omega), \rho_1 & b/2 < x < a/2. \end{cases}$$

We note that η_1 and η_2 are, in general, complex functions of the possibly complex frequency ω.

By Floquet theory[1] the displacement and stress satisfy the quasi-periodic boundary conditions across each cell

$$u(a/2) = u(-a/2)\,e^{iqa} \tag{3}$$

$$\sigma(a/2) = \sigma(-a/2)\,e^{iqa} \tag{4}$$

where q is the wave number.

The interfaces are assumed perfectly bonded so that u and σ are continuous across them

$$u(x_0+) = u(x_0-) \qquad x_0 = \pm b/2 \tag{5}$$

$$\sigma(x_0+) = \sigma(x_0-) \qquad x_0 = \pm b/2. \tag{6}$$

VARIATIONAL PRINCIPLES

(a) *Strain energy principle*

Let us choose a pair of sectionally continuous and differentiable functions u and σ which satisfy the constitutive eqn (2), the quasi-periodic boundary conditions (3) and (4) and continuity condition (5). The solution (u, σ) which also satisfies the equation of motion (1) and the continuity condition (6) is given by the variational equation

$$\delta I(u) + \sigma(a/2)\delta u(a/2)(1 - e^{-2iqa}) = 0 \tag{7}$$

where

$$I(u) = \int_{-a/2}^{a/2} \left\{ -\frac{\eta(x)}{2}\left(\frac{du}{dx}\right)^2 + \frac{\rho(x)\omega^2 u^2}{2} \right\} dx.$$

Proof: Taking variations with respect to u and after carrying out the necessary integrations, we obtain

$$\delta I(u) + \sigma(a/2)\delta u(a/2)(1 - e^{-2iqa}) = \int_{-a/2}^{a/2} \left\{ \frac{d}{dx}\left(\eta(x)\frac{du}{dx}\right) + \rho(x)\omega^2 u \right\} \delta u \, dx$$

$$+ \left\{ \sigma(a/2) - \eta_1 \frac{du}{dx}(a/2) \right\} \delta u(a/2) + \eta_1 \frac{du}{dx}(-a/2)\delta u(-a/2) - \sigma(a/2)\delta u(a/2)\,e^{-2iqa}$$

$$+ \left\langle \eta(b/2)\frac{du}{dx}(b/2)\delta u(b/2) \right\rangle + \left\langle \eta(-b/2)\frac{du}{dx}(-b/2)\delta u(-b/2) \right\rangle.$$

Here we use the notation

$$\langle g(x_0)\rangle = g(x_0^+) - g(x_0^-).$$

If u and σ are restricted to the subclass of functions which satisfy eqns (2)–(5), this expression reduces to

$$\int_{-a/2}^{a/2}\left(\frac{d\sigma}{dx}+\rho(x)\omega^2 u\right)\delta u\,dx + \langle\sigma(b/2)\rangle\delta u(b/2)+\langle\sigma(-b/2)\rangle\delta u(-b/2).$$

This expression vanishes for arbitrary δu if and only if u and σ satisfy eqns (1) and (6).

(b) *Complementary energy principle*

Let us choose a pair of sectionally continuous and differentiable functions u and σ which satisfy the equation of motion (1), the quasi-periodic boundary conditions (3) and (4) and continuity condition (6). The solution (u, σ) which also satisfies the constitutive eqn (2) and the continuity condition (5) is given by the variational equation

$$\delta J(\sigma) + u(a/2)\delta\sigma(a/2)(1-e^{-2iqa}) = 0 \tag{8}$$

where

$$J(\sigma) = \int_{-a/2}^{a/2}\left\{-\frac{\sigma^2}{2\eta(x)}+\frac{(\sigma')^2}{2\rho(x)\omega^2}\right\}dx$$

and the prime denotes differentiation with respect to x.

Proof: Taking variations with respect to σ and after carrying out the necessary integrations we obtain

$$\begin{aligned}\delta J(\sigma) + u(a/2)\delta\sigma(a/2)[1-e^{-2iqa}] = &\int_{-a/2}^{a/2} -\left\{\frac{\sigma}{\eta(x)}+\frac{d}{dx}\left(\frac{\sigma'}{\rho(x)\omega^2}\right)\right\}\delta\sigma\,dx\\ &+\left(u(a/2)+\frac{\sigma'(a/2)}{\rho_1\omega^2}\right)\delta\sigma(a/2)\\ &-\frac{\sigma'(-a/2)}{\rho_1\omega^2}\delta\sigma(-a/2) - u(a/2)\delta\sigma(a/2)\,e^{-2iqa}\\ &-\left\langle\frac{\sigma'(b/2)}{\rho(b/2)\omega^2}\delta\sigma(b/2)\right\rangle\\ &-\left\langle\frac{\sigma'(-b/2)}{\rho(-b/2)\omega^2}\delta\sigma(-b/2)\right\rangle.\end{aligned}$$

If u and σ are restricted to the subclass of functions which satisfy eqns (1), (3), (4) and (6), this expression reduces to

$$\int_{-a/2}^{a/2}\left(\frac{du}{dx}-\frac{\sigma}{\eta(x)}\right)\delta\sigma\,dx + \langle u(b/2)\rangle\delta\sigma(b/2)+\langle u(-b/2)\rangle\delta\sigma(-b/2).$$

This expression vanishes for arbitrary $\delta\sigma$ if and only if u and σ satisfy eqns (2) and (5).

(c) *Hellinger–Reissner principle*

Let us choose a pair of sectionally continuous and differentiable functions u and σ which satisfy the quasi-periodic boundary conditions (3) and (4) and continuity condition (5). The solution (u, σ) which also satisfies the equation of motion (1), the constitutive eqn (2) and continuity condition (6) is given by the variational equation

$$\delta K(u,\sigma)+\sigma(a/2)\delta u(a/2)(1-\mathrm{e}^{-2iqa})=0 \tag{9}$$

where

$$K(u,\sigma)=\int_{-a/2}^{a/2}\left(-\sigma\frac{\mathrm{d}u}{\mathrm{d}x}+\frac{\sigma^2}{2\eta(x)}+\frac{\rho(x)\omega^2u^2}{2}\right)\mathrm{d}x.$$

Proof: Taking variations with respect to u and σ and after carrying out the necessary integrations we obtain

$$\begin{aligned}\delta K(u,\sigma)+\sigma(a/2)\delta u(a/2)(1-\mathrm{e}^{-2iqa})=&\int_{-a/2}^{a/2}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}x}+\rho(x)\omega^2u\right)\delta u\,\mathrm{d}x+\int_{-a/2}^{a/2}\left(\frac{\sigma}{\eta(x)}-\frac{\mathrm{d}u}{\mathrm{d}x}\right)\delta\sigma\,\mathrm{d}x\\&+\sigma(-a/2)\delta u(-a/2)-\sigma(a/2)\delta u(a/2)\,\mathrm{e}^{-2iqa}\\&+\langle\sigma(b/2)\delta u(b/2)\rangle+\langle\sigma(-b/2)\delta u(-b/2)\rangle.\end{aligned}$$

If u and σ are restricted to the subclass of functions which satisfy eqns (3)–(5), this expression reduces to

$$\int_{-a/2}^{a/2}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}x}+\rho(x)\omega^2u\right)\delta u\,\mathrm{d}x+\int_{-a/2}^{a/2}\left(\frac{\sigma}{\eta(x)}-\frac{\mathrm{d}u}{\mathrm{d}x}\right)\delta\sigma\,\mathrm{d}x+\langle\sigma(b/2)\rangle\delta u(b/2)+\langle\sigma(-b/2)\rangle\delta u(-b/2).$$

This expression vanishes for arbitrary δu and $\delta\sigma$ if and only if u and σ satisfy eqns (1), (2) and (6).

(d) *A general variational principle*

The variational equation

$$\delta[K(u,\sigma)+L_b+L_{-b}]+\sigma(a/2)\{\delta u(a/2)-\delta u(-a/2)\,\mathrm{e}^{-iqa}\}+\{u(a/2)\,\mathrm{e}^{-iqa}-u(-a/2)\}\delta\sigma(a/2)=0 \tag{10}$$

where

$$L_b=-\tfrac{1}{2}[\sigma(b^-/2)+\sigma(b^+/2)]\langle u(b/2)\rangle$$
$$L_{-b}=-\tfrac{1}{2}[\sigma(-b^-/2)+\sigma(-b^+/2)]\langle u(-b/2)\rangle$$

and $K(u,\sigma)$ is defined in eqn (9) is completely equivalent to the original boundary-value problem (1)–(6).

Proof: Taking variations with respect to u and σ and after carrying out the necessary integrations, we obtain

$$\begin{aligned}&\int_{-a/2}^{a/2}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}x}+\rho(x)\omega^2u\right)\delta u\,\mathrm{d}x+\int_{-a/2}^{a/2}\left(\frac{\sigma}{\eta(x)}-\frac{\mathrm{d}u}{\mathrm{d}x}\right)\delta\sigma\,\mathrm{d}x+\{u(a/2)\,\mathrm{e}^{-iqa}-u(-a/2)\}\delta\sigma(a/2)\\&\quad-\{\sigma(a/2)\,\mathrm{e}^{-iqa}-\sigma(-a/2)\}\delta u(-a/2)-\tfrac{1}{2}\langle u(b/2)\rangle\{\delta\sigma(b^+/2)+\delta\sigma(b^-/2)\}-\tfrac{1}{2}\langle u(-b/2)\rangle\\&\quad\times\{\delta\sigma(-b^+/2)+\delta\sigma(-b^-/2)\}+\tfrac{1}{2}\langle\sigma(b/2)\rangle\{\delta u(b^+/2)+\delta u(b^-/2)\}+\tfrac{1}{2}\langle\sigma(-b/2)\rangle\\&\quad\times\{\delta u(-b^+/2)+\delta u(-b^-/2)\}.\end{aligned}$$

This expression vanishes for arbitrary δu and $\delta\sigma$ if and only if u and σ satisfy the complete set of eqns (1)–(6).

A NUMERICAL EXAMPLE

(a) *The problem and method of solution*

As an illustration, we consider an application of the extended version of the Hellinger–Reissner principle (9). This principle offers greater flexibility than the strain energy or comwhere

$$I_{jk}^u = \int_{-1/2}^{1/2} \frac{\rho(X)}{\rho_1} \frac{\Omega}{2} U_j(X) U_k(X)\,dX$$
$$= \frac{\Omega[(-1)^{j+k} \sin Q + (\theta - 1) \sin \{\alpha(Q + \pi(j+k))\}]}{2(Q + \pi(j+k))} \quad \text{if} \quad Q \neq -\pi(j+k)$$
$$= \frac{\Omega(1 + \alpha(\theta - 1))}{2} \quad \text{if} \quad Q = -\pi(j+k)$$

$$I_{jk}^s = \int_{-1/2}^{1/2} \frac{\eta_2}{2\eta(X)} S_j(X) S_k(X)\,dX$$
$$= \frac{\sin \{\alpha(Q + \pi(j+k))\} + \xi[(-1)^{j+k} \sin Q - \sin \{\alpha(Q + \pi(j+k))\}]}{2(Q + \pi(j+k))} \quad \text{if} \quad Q \neq -\pi(j+k)$$
$$= \frac{\alpha + \xi(1 - \alpha)}{2} \quad \text{if} \quad Q = -\pi(j+k)$$

$$I_{jk}^{su} = \int_{-1/2}^{1/2} \frac{dU_j(X)}{dX} S_k(X)\,dX$$
$$= \frac{(-1)^{j+k} i(Q + 2\pi j) \sin Q}{Q + \pi(j+k)} \quad \text{if} \quad Q \neq -\pi(j+k)$$
$$= i(Q + 2\pi j) \quad \text{if} \quad Q = -\pi(j+k)$$
$$J_{jk}^{su} = (1 - e^{-2iQ})\, e^{i(Q + \pi(j+k))}.$$

Taking variations and equating the coefficients of δC_j and δD_k to zero, we obtain the matrix equation

$$\left[\begin{array}{c|c} 2\mathbf{I}^u & \mathbf{J}^{su} - \mathbf{I}^{su} \\ \hline -(\mathbf{I}^{su})^T & 2\mathbf{I}^s \end{array}\right] \left[\begin{array}{c} \mathbf{C} \\ \hline \mathbf{D} \end{array}\right] = \left[\begin{array}{c} \mathbf{0} \\ \hline \mathbf{0} \end{array}\right].$$

The matrices $\mathbf{I}^u$, $\mathbf{I}^s$, $\mathbf{J}^{su}$ and $\mathbf{I}^{su}$ are each of order $m \times m$ (where $m = 2n + 1$). The elements of $\mathbf{I}^u$ and $\mathbf{I}^s$ are functions of the complex frequency ω through Ω and η. Thus, for any given wave number Q, the frequency ω can be obtained by solving the equation

$$\det [\mathbf{N}(\omega, q)] = 0 \tag{16}$$

where $\mathbf{N}$ is a $2m \times 2m$ matrix

$$\mathbf{N} \equiv \left[\begin{array}{c|c} 2\mathbf{I}^u & \mathbf{J}^{su} - \mathbf{I}^{su} \\ \hline -(\mathbf{I}^{su})^T & 2\mathbf{I}^s \end{array}\right].$$

Once an eigenvalue ω is known, the corresponding eigenvector and hence the mode shapes U and S can be easily calculated.

In order to proceed further, we must assume a functional form for $\xi(\omega)$. We assume, as in [8] that the matrix is a standard linear solid and the filament is elastic, so that

$$\eta_1(s) = \frac{\eta_1{}^R + s\tau_1\eta_1{}^G}{1 + s\tau_1}, \quad \eta_2(s) = \eta_2{}^G \tag{17}$$

where $s = i\omega$, $\eta_1{}^R$ and $\eta_1{}^G$ are the "rubbery" (at long time) and "glassy" (at short time) moduli respectively and τ_1 is the relaxation time of the matrix material. The filament is elastic so that η_2 is independent of ω and is real.

The standard linear solid is used as an example. The method applies to any linear viscoelastic material as long as $\eta_1(\omega)$ and $\eta_2(\omega)$ are known.

We define some further dimensionless quantities

$$\eta^G = \eta_2{}^G/\eta_1{}^G, \quad \delta_1 = \eta_1{}^R/\eta_1{}^G$$
$$t_1 = C_1{}^G\tau_1/a, \quad s' = sa/C_1{}^G \tag{18}$$
$$\hat{\omega} = -is'\left[\frac{1-(1-\theta)\alpha}{1-(1-\eta^G)\alpha}\right]^{1/2}$$

where $C_1{}^G = \sqrt{(\eta_1{}^G/\rho_1)}$ is a reference phase velocity.

In terms of these

$$\xi(s) = \frac{\eta^G(1+s't_1)}{(\delta_1+s't_1)}$$
$$\Omega(s) = -(s')^2/\eta^G. \tag{19}$$

We now rewrite the matrix $\mathbf{N}(s)$ in order to show the frequency dependence explicitly. Thus, eqn (16) becomes

$$\det\left[\begin{array}{c|c} \mathbf{A}s'^2 & \mathbf{B} \\ \hline -\mathbf{E} & \mathbf{G}+\mathbf{F}\dfrac{(1+s't_1)}{(\delta_1+s't_1)} \end{array}\right] = 0 \tag{20}$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{E}$, $\mathbf{F}$, $\mathbf{G}$ are $(m \times m)$ matrices given by

$$A_{jk} = -\frac{[(-1)^{j+k}\sin Q + (\theta-1)\sin\{\alpha(Q+\pi(j+k))\}]}{\eta^G(Q+\pi(j+k))} \quad \text{if} \quad Q \neq -\pi(j+k)$$
$$= -\frac{(1+\alpha(\theta-1))}{\eta^G} \quad \text{if} \quad Q = -\pi(j+k)$$
$$B_{jk} = J_{jk}^{su} - I_{jk}^{su}$$
$$E_{jk} = I_{kj}^{su}$$
$$F_{jk} = \frac{\eta^G[(-1)^{j+k}\sin Q - \sin\{\alpha(Q+\pi(j+k))\}]}{Q+\pi(j+k)} \quad \text{if} \quad Q \neq \pi(j+k)$$
$$= \eta^G(1-\alpha) \quad \text{if} \quad Q = -\pi(j+k)$$
$$G_{jk} = \frac{\sin\{\alpha(Q+\pi(j+k))\}}{Q+\pi(j+k)} \quad \text{if} \quad Q \neq -\pi(j+k)$$
$$= \alpha \quad \text{if} \quad Q = -\pi(j+k).$$

The elements of $\mathbf{A}$, $\mathbf{B}$, $\mathbf{E}$, $\mathbf{F}$ and $\mathbf{G}$ are known once the wave number and the other parameters of the problem are specified. Equation (20) represents a nonlinear eigenvalue problem since many of the elements of $\mathbf{N}$ are nonlinear functions of s. It is possible to reduce this to a standard eigenvalue problem so that s' is an eigenvalue of a new matrix $\mathbf{M}$ of order $3m \times 3m$. This increases the size of the problem but we can now use any of the standard computer algorithms for determination of eigenvalues of a matrix.

We observe that

$$\det\left[\begin{array}{c|c} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \hline \mathbf{M}_{21} & \mathbf{M}_{22} \end{array}\right] = \det[\mathbf{M}_{11}\mathbf{M}_{22} - \mathbf{M}_{11}\mathbf{M}_{21}\mathbf{M}_{11}^{-1}\mathbf{M}_{12}]$$

if the matrices are conformable and $\mathbf{M}_{11}^{-1}$ exists, so that eqns (20) can be expanded as

$$\det[\mathbf{I}s'^3 + \mathbf{P}s'^2 + \mathbf{H}s' + \gamma\mathbf{H}] = 0 \tag{21}$$

where

$$\mathbf{P} = \mathbf{R}^{-1}(\mathbf{F} + \mathbf{G}\delta_1)$$
$$\mathbf{H} = t_1\mathbf{R}^{-1}\mathbf{E}\mathbf{A}^{-1}\mathbf{B}$$
$$\gamma = \delta_1/t_1$$
$$\mathbf{R} = t_1(\mathbf{G} + \mathbf{F}),$$

and $\mathbf{I}$ is the unit matrix.

Now s' is an eigenvalue of a $3m \times 3m$ matrix $\mathbf{M}$ defined as

$$M = \left[\begin{array}{c|c|c} \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{I} \\ \hline -\gamma\mathbf{H} & -\mathbf{H} & -\mathbf{P} \end{array}\right]. \tag{22}$$

This can be easily proved by expanding the equation

$$\det[\mathbf{M} - \mathbf{I}s'] = 0. \tag{23}$$

(b) *Results and conclusions*

Numerical calculations are carried out for the following values of the parameters[8]

$$\eta^G = 4 \text{ and } 50 \qquad \theta = 3$$
$$\delta_1 = 0.70 \qquad t_1 = 0.0455 \qquad \alpha = 0.5$$
$$m = 7 \text{ and } 11.$$

The wave number Q is specified and only the range $0 \le Q \le \pi$ needs to be studied[10] because the Floquet form of the solution is not uniquely determined.

The dispersion curves are obtained by calculating the eigenvalues of $\mathbf{M}$ (eqn 22) using an IBM 360/67 computer. A standard *QR* algorithm is used. Once the frequency is known for a certain Q we calculate the corresponding eigenvector of the matrix $\mathbf{N}$ (eqns 15, 20). Now the mode shape U and S are obtained from eqn (13).

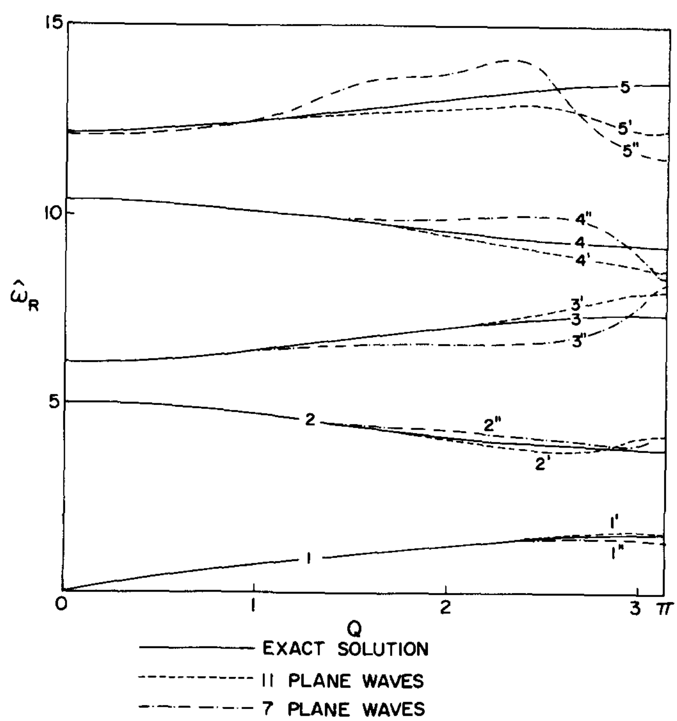

Fig. 2. Dispersion curves $\hat{\omega}_R$ vs Q; $\eta^G = 4$, $\theta = 3$, $\alpha = 0.5$, $\delta_1 = 0.70$, $t_1 = 0.0455$.

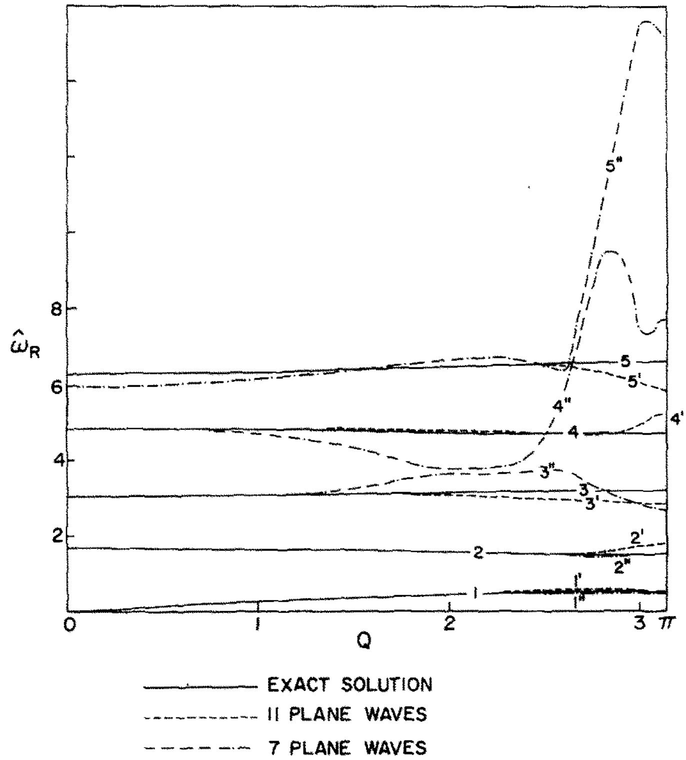

Fig. 3. Dispersion curves $\hat{\omega}_R$ vs Q; $\eta^G = 50$, $\theta = 3$, $\alpha = 0.5$, $\delta_1 = 0.70$, $t_1 = 0.0455$.

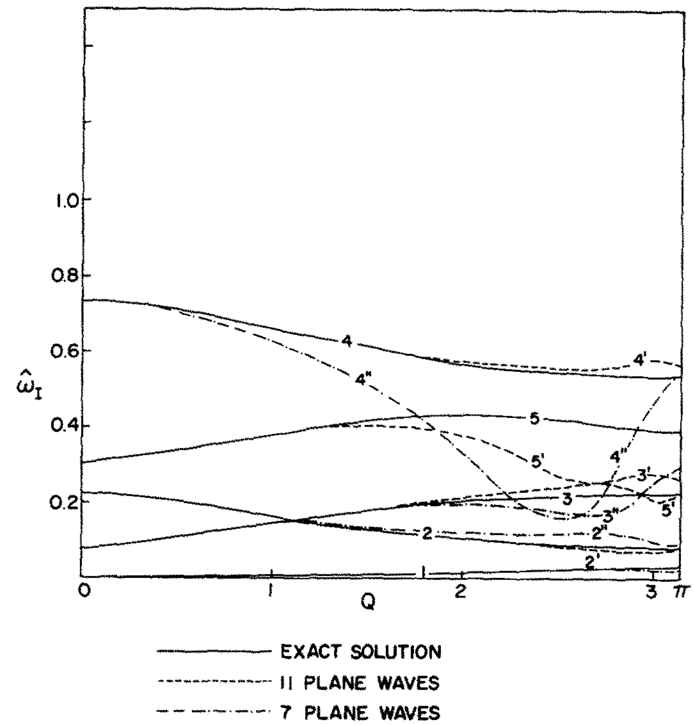

Fig. 4. Damping curves $\hat{\omega}_1$ vs Q; $\eta^G = 4$, $\theta = 3$, $\alpha = 0.5$, $\delta_1 = 0.7$, $t_1 = 0.0455$.

Figures 2–5 show the real and imaginary parts of the frequency $\hat{\omega}$ as a function of the wave number Q for the first 5 modes. The approximate solutions for $m = 7$ and 11 are compared with the exact solution obtained from (11). The agreement is excellent with $m = 11$ for the first two modes but the approximate solutions become progressively inaccurate for higher modes. The

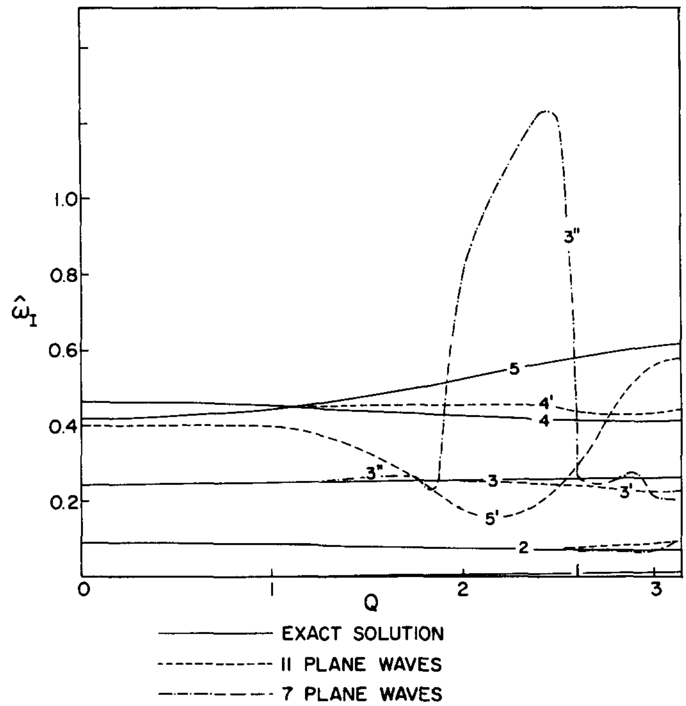

Fig. 5. Damping curves $\hat{\omega}_I$ vs Q; $\eta^G = 50$, $\theta = 3$, $\alpha = 0.5$, $\delta_1 = 0.7$, $t_1 = 0.0455$.

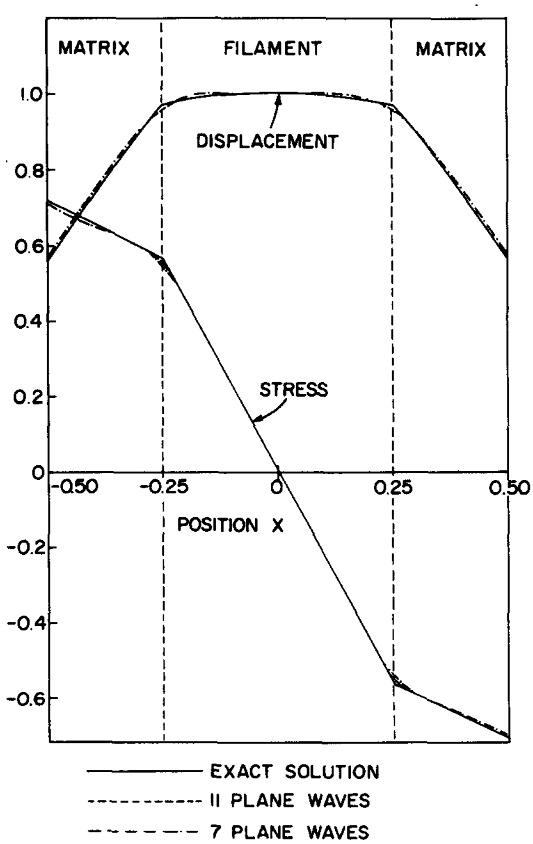

Fig. 6. Displacement and stress distribution for the first mode; $\eta^G = 4$, $\theta = 3$, $\alpha = 0.5$, $Q = \pi/2$.

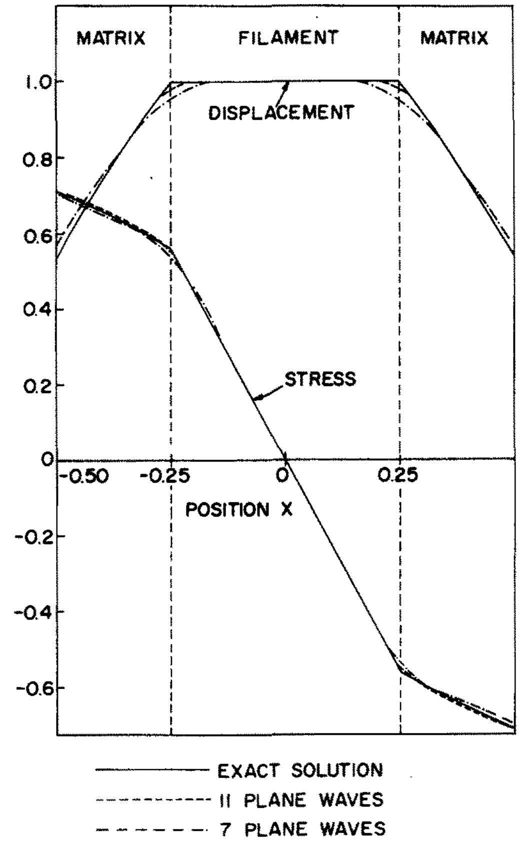

Fig. 7. Displacement and stress distribution for the first mode; $\eta^G = 50$, $\theta = 3$, $\alpha = 0.5$, $Q = \pi/2$.

maximum error in the real part of the frequency for $m = 11$ is of the order of 10% and occurs for the fifth mode for $Q = \pi$. The error is much higher for the imaginary part of the frequency for the fifth mode. We recall that $\hat{\omega}_I$ determines the rate of exponential decay in time.

A comparison with [8] *shows that in this case the finite difference scheme is more efficient.* Also, a comparison with [4] shows that the Hellinger–Reissner principle does not work as efficiently as it does in the elastic case. The fundamental mode, however, is obtained very accurately with $m = 7$ and this mode dominates since the higher modes decay faster.

Mode shapes are shown in Figs. 6 and 7. The real parts of the displacement and stress are shown for the fundamental mode for $Q = \pi/2$. The normalizing constant is chosen as in [8] so that $U(0) = 1 + 0i$. The results for $m = 11$ are in excellent agreement with the exact solution even though the stress and displacement are expanded in series of smooth functions (13) while the solutions contain sharp discontinuities in slope at the interfaces. The stress solution is more accurate than in [8] because here it is obtained independently rather than being obtained from U by numerical differentiation.

The Hellinger–Reissner principle enables us to obtain continuity of both stress and displacement across the interface while if the strain energy principle is used together with a smooth displacement field a stress discontinuity results[1].

To sum up, this paper presents some variational principles which can be used to study wave propagation in laminated viscoelastic composites. A numerical example gives encouraging results. These principles can be extended to higher dimensions to study wave propagation in fiber reinforced viscoelastic composites.

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